Note on the Central Differential Equation in the Relativity Theory of Gravitation.

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The critical equation in Prof. Einstein's relativity theory of gravitation is given in the form*

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2-1}{h^2} + \frac{2mu}{h^2} + 2mu^3,$$

where m is the mass of the sun, u is the reciprocal of the distance of the moving body from the sun,

$$h^2 = ma(1-e^2), \qquad c^2-1 = -m/a,$$

a is the "major semi-axis of the orbit," the units being astronomical, and e is the "excentricity" of the orbit.

The foregoing equation can be integrated exactly, the functions involved being elliptic and not circular; and the approximations for the solar system are easily obtained, because the modulus of the elliptic functions is small. The necessary analysis is as follows.

1. The equation is

$$\begin{aligned}\left(\frac{du}{d\phi}\right)^2 &= -\frac{m}{ah^2} + 2\frac{m}{h^2}u - u^2 + 2mu^3\\ &= \mathrm{A}u^3 + 3\mathrm{B}u^2 + 3\mathrm{C}u + \mathrm{D} = \mathrm{U},\end{aligned}$$

where $\quad \mathrm{A} = 2m, \qquad \mathrm{B} = -\tfrac{1}{3}, \qquad \mathrm{C} = \frac{2}{3}\frac{m}{h^2}, \qquad \mathrm{D} = -\frac{m}{ah^2}.$

The equation $\mathrm{U} = 0$ has three roots; and their character, as to reality, depends upon the sign of $\mathrm{G}^2 + 4\mathrm{H}^3$, where (with the usual notation)

$$\mathrm{H} = \mathrm{AC} - \mathrm{B}^2 = -\frac{1}{9} + \frac{4}{3}\frac{m^2}{h^2},$$

$$\mathrm{G} = \mathrm{A}^2\mathrm{D} - 3\mathrm{ABC} + 2\mathrm{B}^3 = -\frac{2}{27} + \frac{4}{3}\frac{m^2}{h^2} - 4\frac{m}{ah^2}.$$

On substitution and reduction, we have

$$\begin{aligned}\mathrm{G}^2 + 4\mathrm{H}^3 &= \frac{16}{27}\frac{m^3}{ah^2} - \frac{16}{27}\frac{m^4}{h^4} - \frac{32m^5}{3ah^4} + \frac{16m^6}{a^2h^4} + \frac{256}{27}\frac{m^6}{h^6}\\ &= \frac{m^2}{h^2}\left\{-\frac{16}{27}\frac{m^2}{h^2} + \frac{16}{27}\frac{m}{a} - \frac{32}{3}\frac{m}{a}\frac{m^2}{h^2} + 16\frac{m^2}{a^2}\frac{m^2}{h^2} + \frac{256}{27}\frac{m^6}{a^6}\right\}.\end{aligned}$$

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* Prof. Eddington's 'Report on the Relativity Theory of Gravitation,' 2nd edition, p. 50 (1920).

By using the relation

$$h^2 = ma(1-e^2),$$

the last expression leads to the equation

$$\mathrm{G}^2+4\mathrm{H}^3 = \frac{m^4}{h^4}\left\{-\frac{16}{27}e^2+\left(\frac{400}{27}-\frac{64}{3}e^2+16e^4\right)\frac{m^4}{h^4}\right\}.$$

In the solar system, as regards even the nearest known planets, m^2/h^2 is of the order 10^{-8}, while e^2 is never of so small an order; consequently, the first term within the bracket governs the sign of $\mathrm{G}^2+4\mathrm{H}^3$. Thus $\mathrm{G}^2+4\mathrm{H}^3$ is negative, but not attaining a limit zero within the system; therefore, by the usual rule, the three roots of $\mathrm{U} = 0$ are real, and no two of them are equal. Let them be denoted by α, β, γ, arranged in descending order of magnitude; their product is $1/2ah^2$, a positive quantity; hence either all three real roots must be positive, or one of them must be positive and the other two must be negative.

2. As a matter of fact, the three roots are positive for the solar system. Let

$$\mathrm{A}u+\mathrm{B} = t;$$

then

$$t^3+3\mathrm{H}t+\mathrm{G} = 0.$$

Comparing this equation, as usual when the three roots are real, with the equation

$$\cos^3\theta-\tfrac{3}{4}\cos\theta-\tfrac{1}{4}\cos 3\theta = 0,$$

we find

$$t = 2(-\mathrm{H})^{\frac{1}{2}}\cos\theta,$$

where

$$\cos 3\theta = -\tfrac{1}{2}\,\frac{\mathrm{G}}{(-\mathrm{H}^3)^{\frac{1}{2}}},$$

giving a positive value for $\cos 3\theta$. We shall assume a positive value for 3θ, between 0 and $\frac{1}{2}\pi$; there is no loss of generality in the assumption. The three values of t correspond to the values θ, $\theta+\frac{2}{3}\pi$, $\theta+\frac{4}{3}\pi$; hence, as

$$\cos(\theta+\tfrac{2}{3}\pi) = -\tfrac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta,$$

$$\cos(\theta+\tfrac{4}{3}\pi) = -\tfrac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta,$$

we have

$$2m\alpha-\tfrac{1}{3} = 2(-\mathrm{H})^{\frac{1}{2}}\cos\theta,$$

$$2m\beta-\tfrac{1}{3} = 2(-\mathrm{H})^{\frac{1}{2}}\cos(\theta+\tfrac{4}{3}\pi),$$

$$2m\gamma-\tfrac{1}{3} = 2(-\mathrm{H})^{\frac{1}{2}}\cos(\theta+\tfrac{2}{3}\pi).$$

Again,

$$\sin^2 3\theta = 1+\frac{\mathrm{G}^2}{4\mathrm{H}^3} = 108\,\frac{m^4e^2}{h^4}$$

approximately, for the solar system; so that

$$\theta = 2\sqrt{3}\,\frac{m^2 e}{h^2}$$

approximately. Also, approximately,

$$(-\mathrm{H})^{\frac{1}{2}} = \tfrac{1}{3} - 2\,\frac{m^2}{h^2};$$

and therefore

$$2m\beta - \tfrac{1}{3} = 2\left(\tfrac{1}{3} - 2\,\frac{m^2}{h^2}\right)\left(-\tfrac{1}{2} + \frac{\sqrt{3}}{2}\,\theta\right)$$

$$= -\tfrac{1}{3} + 2\,\frac{m^2}{h^2} + \frac{\theta}{\sqrt{3}},$$

approximately. Thus β is positive. Consequently γ is positive, and its value is given by

$$2m\gamma - \tfrac{1}{3} = -\tfrac{1}{3} + 2\,\frac{m^2}{h^2} - \frac{\theta}{\sqrt{3}}.$$

3. Returning now to the original differential equation, we have

$$\frac{du}{d\phi} = \pm\,\mathrm{U}^{\frac{1}{2}}$$

$$= \pm\,\{2m(u-\alpha)(u-\beta)(u-\gamma)\}^{\frac{1}{2}}.$$

Now $du/d\phi$ is real. For the solar system, $2m\alpha$ is approximately equal to 1, while $m = 1{\cdot}47$, and u is of order not higher than 10^{-8}. Hence

$$\beta \geqq u \geqq \gamma,$$

β being the value of u at perihelion. We shall assume that ϕ is measured positively throughout the orbit, so that $d\phi$ is positive. From perihelion to aphelion u decreases, so that du is negative in this range. Thus $du/d\phi$ is negative, and therefore

$$\frac{du}{d\phi} = -\{2m(u-\alpha)(u-\beta)(u-\gamma)\}^{\frac{1}{2}}.$$

Substitute

$$u = \gamma + (\beta-\gamma)\sin^2\psi;$$

after a slight reduction, we find

$$d\phi = -2\rho\,\frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}},$$

where $\rho = \{2m(\alpha-\gamma)\}^{-\frac{1}{2}}, \qquad k^2 = \dfrac{\beta-\gamma}{\alpha-\gamma}.$

The value of u at perihelion is β, and it subsequently decreases; hence the value of ψ at perihelion is $\frac{1}{2}\pi$, and it subsequently decreases. Consequently, at any place before aphelion in the orbit, we have

$$\begin{aligned} \frac{\phi-\varpi}{2\rho} &= -\int_{\frac{1}{2}\pi}^{\psi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} \\ &= \int_{\psi}^{\frac{1}{2}\pi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} \\ &= \int_0^{\frac{1}{2}\pi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} - \int_0^{\psi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} \\ &= \mathrm{K}-\mu, \end{aligned}$$

where $\phi = \varpi$ at perihelion, and, in the usual notation of elliptic functions, K is the complete elliptic integral of the first kind, and

$$\psi = am\,\mu,$$

that is,

$$\psi = am\left(\mathrm{K}-\frac{\phi-\varpi}{2\rho}\right).$$

Now

$$\begin{aligned} u &= \gamma+(\beta-\gamma)\sin^2\psi \\ &= \gamma+(\beta-\gamma)\,sn^2\,\mu \\ &= \gamma+(\beta-\gamma)\,\frac{1-cn\,2\mu}{1+dn\,2\mu} \\ &= \gamma+(\beta-\gamma)\,\frac{1-cn\left(2\mathrm{K}-\frac{\phi-\varpi}{\rho}\right)}{1+dn\left(2\mathrm{K}-\frac{\phi-\varpi}{\rho}\right)} \\ &= \gamma+(\beta-\gamma)\,\frac{1+cn\,\frac{\phi-\varpi}{\rho}}{1+dn\,\frac{\phi-\varpi}{\rho}}, \end{aligned}$$

which is the accurate equation of the orbit represented by the original differential equation. It involves elliptic functions, instead of the circular functions that occur in the equation of the Newtonian orbit.

4. Further, the value of ψ (decreasing from $\frac{1}{2}\pi$ at perihelion) is zero at aphelion; so, if ϕ_1 denote the value of ϕ at aphelion, we have

$$\phi_1-\varpi = 2\rho\mathrm{K}.$$

From aphelion to perihelion we have similar relations; so that, if at the next perihelion, ϕ_2 be the value of ϕ, which is to begin with ϕ_1 at aphelion, we have

$$\phi_2-\phi_1 = 2\rho\mathrm{K}.$$

Thus

$$\phi_2-\varpi = 4\rho\mathrm{K}.$$

Let the advance of the perihelion in one revolution be denoted by Δ; then

$$\Delta = (\phi_2 - \varpi) - 2\pi = 4\rho\mathrm{K} - 2\pi.$$

The expression is accurate, and not merely approximate, in connection with the original differential equation so long as the magnitudes in that equation allow three real roots for $\mathrm{U} = 0$, whatever be the gravitational system.

The preceding results are accurate, and not merely approximate, subject to the condition just stated. In the exposition of the application to the solar system, and specially with reference to the advance of the perihelion of Mercury in one complete revolution, approximation is made with regard to the magnitudes of the solar system that are involved. To bring the approximation into relation with the foregoing accurate results, we must deal with the two relations

$$u = \gamma + (\beta - \gamma)\frac{1 + cn\dfrac{\phi - \varpi}{\rho}}{1 + dn\dfrac{\phi - \varpi}{\rho}}, \qquad \Delta = 4\rho\mathrm{K} - 2\pi.$$

5. As regards the approximation to Δ, we require the most important terms in ρ and in K.

The value of ρ is

$$\{2m(\alpha - \gamma)\}^{-\frac{1}{2}}.$$

From the foregoing values of α, β, γ, we have

$$\begin{aligned} 2m(\alpha - \gamma) &= 2(-\mathrm{H})^{\frac{1}{2}}\{\cos\theta - \cos(\theta + \tfrac{2}{3}\pi)\} \\ &= 2(-\mathrm{H})^{\frac{1}{2}}\left(\frac{3}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) \\ &= \frac{2}{3}\left(1 - 6\frac{m^2}{h^2}\right)\frac{3}{2}\left(1 + \frac{\theta}{\sqrt{3}}\right) \\ &= 1 + \frac{\theta}{\sqrt{3}} - 6\frac{m^2}{h^2}, \end{aligned}$$

to the order of approximation that includes m^2/h^2 but no higher power of m/h. Hence, to this order,

$$\begin{aligned} \rho &= 1 + 3\frac{m^2}{h^2} - \frac{1}{2\sqrt{3}}\theta \\ &= 1 + 3\frac{m^2}{h^2} - \frac{m^2 e}{h^2}. \end{aligned}$$

The value of k^2 is

$$\frac{\beta - \gamma}{\alpha - \gamma}.$$

Again from the foregoing values of β and γ, we have

$$2m(\beta - \gamma) = 2\frac{\theta}{\sqrt{3}} = 4\frac{m^2 e}{h^2},$$

to the order retained; and therefore, to this order,

$$k^2 = 4\frac{m^2}{h^2}e.$$

But
$$K = \int_0^{\frac{1}{2}\pi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}}$$
$$= \int_0^{\frac{1}{2}\pi} (1+\tfrac{1}{2}k^2\sin^2\psi)\,d\psi$$

to the order of approximation: that is, to the order,

$$K = \tfrac{1}{2}\pi(1+\tfrac{1}{4}k^2)$$
$$= \tfrac{1}{2}\pi\left(1+\frac{m^2}{h^2}e\right).$$

Thus
$$\Delta = 4\rho K - 2\pi$$
$$= 2\pi\left(1+3\frac{m^2}{h^2}-\frac{m^2e}{h^2}\right)\left(1+\frac{m^2}{h^2}e\right)-2\pi$$
$$= 2\pi\,.\,3\frac{m^2}{h^2};$$

and therefore there follows Prof. Einstein's inference from his theory that, to the order of approximation adopted, the perihelion of the moving body advances a fraction of a revolution equal to

$$3\frac{m^2}{h^2},$$

during each revolution.

6. As regards the approximation to u, we write

$$\phi-\varpi = \sigma, \qquad x = \frac{\pi\sigma}{2\rho K};$$

and as k^2 is small, the quantity q of the elliptic functions is

$$q = \frac{1}{16}k^2.$$

Now from the expression for $cn\,\frac{\sigma}{\rho}$, we have

$$cn\,\frac{\sigma}{\rho} = (1-4q\sin^2 x)\cos x,$$

so that we want $\cos x$. But

$$\frac{2\rho K}{\pi} = 1+3\frac{m^2}{h^2},$$

and therefore

$$\cos x = \cos\left\{\left(1-3\frac{m^2}{h^2}\right)\sigma\right\}$$
$$= \cos\sigma + 3\frac{m^2}{h^2}\sigma\sin\sigma,$$

to the order retained, so that,

$$cn\,\frac{\sigma}{\rho} = \cos\sigma + 3\,\frac{m^2}{h^2}\,\sigma\sin\sigma - \frac{m^2}{h^2}\,e\sin^2\sigma\cos\sigma.$$

Again, from the expression for $dn\,\frac{\sigma}{\rho}$, we have

$$\begin{aligned} dn\,\frac{\sigma}{\rho} &= 1-8q\sin^2 x \\ &= 1-2\,\frac{m^2}{h^2}\,e\sin^2\sigma. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{1+cn\,\sigma/\rho}{1+dn\,\sigma/\rho} &= \tfrac{1}{2}\,\frac{1+\cos\sigma+3m^2/h^2\,\sigma\sin\sigma - m^2/h^2\,e\sin^2\sigma\cos\sigma}{1-m^2/h^2\,e\sin^2\sigma} \\ &= \tfrac{1}{2}\left(1+\cos\sigma+3\,\frac{m^2}{h^2}\,\sigma\sin\sigma+\frac{m^2}{h^2}\,e\sin^2\sigma\right) \end{aligned}$$

to our order of approximation. Also

$$2m\gamma = 2\,\frac{m^2}{h^2} - 2\,\frac{m^2}{h^2}\,e,$$

so that

$$\gamma = \frac{m}{h^2}(1-e);$$

and

$$2m(\beta-\gamma) = 2\,\frac{\theta}{\sqrt{3}} = 4\,\frac{m^2 e}{h^2},$$

so that

$$\beta-\gamma = 2\,\frac{m}{h^2}\,e.$$

Thus

$$\begin{aligned} u &= \frac{m}{h^2}(1-e) + \frac{m}{h^2}\,e\left(1+\cos\sigma+3\,\frac{m^2}{h^2}\,\sigma\sin\sigma+\frac{m^2}{h^2}\,e\sin^2\sigma\right) \\ &= \frac{m}{h^2}\left\{1+e\cos\sigma+3\,\frac{m^2 e}{h^2}\,\sigma\sin\sigma+\frac{m^2}{h^2}\,e^2\sin^2\sigma\right\} \\ &= \frac{m}{h^2}\left\{1+e\cos(\phi-\varpi)+\frac{m^2}{h^2}\,e^2\sin^2(\phi-\varpi)\right. \\ &\qquad\qquad \left. +3\,\frac{m^2 e}{h^2}(\phi-\varpi)\sin(\phi-\varpi)\right\}. \end{aligned}$$

7. If a closer approximation is desired for any body in which the quantity m^2/h^2 is not of so small an order as in the known solar system, fuller values of the roots α, β, γ of $U = 0$ should be found, and can easily be found. But an inspection of the value of G^2+4H^3, upon which these roots depend, shows that, for any known bodies, the further approximations are ordinarily not worthy of account.